Quantization on symplectic symmetric spaces

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Symplectic symmetric manifolds G/H with G simple are divided into four classes [17]: (a) Hermitian symmetric spaces; (b) semi-Kählerian irreducible symmetric spaces; (c) para-Hermitian symmetric spaces of the first category; (d) para-Hermitian symmetric spaces of the second category. The spaces of the three latter classes are not Riemannian, and each has a Riemannian form belonging to the class of Hermitian symmetric spaces.

Berezin constructed quantization on spaces of class (a). We would like to outline a program for a quantization in the spirit of Berezin for other classes of symplectic homogeneous manifolds. In these lectures we restrict ourselves to class (c). The local classification of spaces of class (c) is given in §3.

There is an inspiring analogy between (a) and (c), which starts at the coordinate level: $z, \bar{z} \longleftrightarrow \xi, \eta$, see §3, and continues on the level of formulae and so on. On the other hand, it is well-known, that the passage from the Riemannian case to the non-Riemannian one drastically increases the difficulties. So, in this theory there are still many interesting open problems.

§1. Quantizations on the plane

First we have classical mechanics. Let us consider a particle with one degree of freedom, say, a linear oscilator with the coordinate q and the impulse p. Then the energy is $H = (1/2)p^2 + q^2$. To pass to quantum mechanics we replace functions by operators:

$$p \to \hat{p} = \frac{h}{i}\frac{d}{dq}, \; q \to \hat{q} = q$$

These operators act in $L^2(\mathbb{R})$. Then the energy H goes to the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + \hat{q}^2 = -\frac{h^2}{2}\frac{d^2}{dq^2} + q^2$$

In a general sense, quantization is the passage from functions to operators. Returning to our particle, we come across the problem : what operator has to be assigned to a function more complicated than p, q, H, for example, pq, p^2q^3 etc.? The point is that the operators $\hat{p}$ and $\hat{q}$ do not commute: $[\hat{p}, \hat{q}] = h/i$, while the functions p and q do it. This problem can be solved in different ways. First let us consider qp-quantization. In any monomial

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we write q from left of p and then we put the huts. We obtain the correspondence $A(q,p) \to \hat{A}$ for polynomials, and we can extend it on arbitrary functions. The function $A(p,q)$ is called *qp-symbol* of the operator $\hat{A}$. There is a simple way to find $A(p,q)$ for given $\hat{A}$. Denote

$$\Phi_p(q) = \Phi(q,p) = exp\frac{ipq}{h}$$

Then

$$A(q,p) = \frac{\hat{A}\Phi_p(q)}{\Phi_p(q)}$$

Therefore the kernel $K(q,v)$ of an operator $\hat{A}$ is expressed in term of its qp-symbol as follows:

$$K(q,v) = \frac{1}{2\pi h}\int A(q,t)\frac{\Phi(q,t)}{\Phi(v,t)}dt$$

(all integrals are taken over $\mathbb{R}$ etc.) The multiplication of operators gives rise to a multiplication of symbols:

$$(A*B)(q,p) = \int A(q,t)B(s,p)\mathcal{B}(q,p;s,t)ds\,dt$$

where

$$\mathcal{B}(q,p;s,t) = \frac{1}{2\pi h}\cdot\frac{\Phi(q,t)\Phi(s,p)}{\Phi(q,p)\Phi(s,t)}$$

There is another expression for the multiplication of symbols:

$$(A*B)(q,p) = \exp(-ih\frac{\partial^2}{\partial t\partial s})A(q,t)B(s,p)\Big|_{t=p,s=q}$$

so that

$$A*B = AB - ih\frac{\partial A}{\partial p}\frac{\partial B}{\partial q} + \frac{(ih)^2}{2!}\frac{\partial^2 A}{\partial p^2}\frac{\partial^2 B}{\partial q^2} + \dots \quad (1.1)$$

From (1.1) we have

$$\lim_{h\to 0} A*B = AB \quad (1.2)$$

$$\lim_{h\to 0}\frac{i}{h}(A*B - B*A) = \{A,B\} \quad (1.3)$$

where $\{A,B\}$ is the Poisson bracket. Equalities (1.2) and (1.3) mean that for qp-quantization the correspondence principle holds.

Similarly we consider pq-quantization (in monomials p stands before q). The kernel $L(q,v)$ of an operator $\hat{A}$ is expressed in terms of its pq-symbol $A(q,p)$ as follows:

$$L(q,v) = \frac{1}{2\pi h}\int A(v,t)\frac{\Phi(q,t)}{\Phi(v,t)}dt$$

The multiplication of pq-symbols is given by formulae:

$$(A*B)(q,p) = \int A(s,p)B(q,t)\mathcal{B}(s,p;q,t)ds\,dt$$

$$= \exp(ih\frac{\partial^2}{\partial t\partial s})B(q,t)A(s,p)\Big|_{t=p,s=q}$$

In this case the correspondence principle (1.2), (1.3) holds too.

There is a connection between pq-symbol $\mathring{A}$ and qp-symbol A of an operator $\hat{A}$ (given by $\mathring{A} \to \hat{A} \to A$):

$$A(q,p) = \int \mathcal{B}(s,p;q,t)B(q,t)ds\,dt$$

and also a connection between operators $\hat{A}$ and $\hat{A}_1$ for which $A(q,p)$ is qp- and pq-symbol respectively: $L(q,v) = K(v,-q+2v)$, i.e. variables q,v are transformed by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

There are also other quantizations: Wick, anti-Wick, Weyl etc., see, for example [5].

§2. Berezin quantization

Recall the concept of quantization proposed by Berezin, see [1–5]. We shall not give it in its full generality, but restrict ourselves to a rather simplified version.

Let M be a symplectic manifold. Then $C^\infty(M)$ is a Lie algebra with respect to the Poisson bracket $\{A,B\}, A,B \in C^\infty(M)$.

Quantization in the sense of Berezin consists of the following two steps:

(I) To construct a family $\mathcal{A}_h$ of associative algebras contained in $C^\infty(M)$ and depending on a parameter $h > 0$ (called the Planck constant), with a multiplication denoted by $*$ (depending on h also). These algebras must satisfy the conditions (a) through (d):

$$(a)\lim_{h\to 0} A_1 * A_2 = A_1A_2;$$

$$(b)\lim_{h\to 0}\frac{i}{h}(A_1 * A_2 - A_2 * A_1) = \{A_1, A_2\} \qquad (2.1);$$

(c) the function $A_0 \equiv 1$ is the unit element of each algebra $\mathcal{A}_h$;

(d) the complex conjugation $A \mapsto \bar{A}$ is an anti-involution of any $\mathcal{A}_h$;

where the multiplication on the right-hand side of (a) is the pointwise multiplication and conditions (a) and (b) together are called the *correspondence principle* (CP).

(II) To construct representations $A \mapsto \hat{A}$ of the algebras $\mathcal{A}_h$ by operators in a Hilbert space.

Berezin mainly considered the case when M is Kählerian, hence has a complex structure. The functions in question are functions $A(z,\bar{z})$ analytic on z and $\bar{z}$ separately. In this case complex conjugation reduces to the permutation of z and $\bar{z}$: $\overline{f(z,\bar{z})} = f(\bar{z},z)$.

For our theory we shall slightly change some of the conditions above: namely, the factor i in (2.1) has to be omitted, the anti-involution is the permutation of arguments, and, finally, we give up the Hilbert space structure of the representation space.

§3. Para-Hermitian symmetric spaces of the first category

Let us recall some facts about semisimple symmetric spaces G/H. Here G is a connected semisimple Lie group with an involutive automorphism $\sigma \neq 1$. Denote by G^σ the subgroup of fixed points of σ. Then H is an open subgroup of G^σ. There exists a Cartan involution τ of G commuting with σ. Let $K \doteq G^\tau$. For Lie groups $G, \ldots$ we denote their Lie algebras by the corresponding small Gothic letters $\mathfrak{g}, \ldots$. We assume that the pair $(\mathfrak{g}, \mathfrak{h})$ is effective, i.e. $\mathfrak{h}$ contains no non-trivial ideals of $\mathfrak{g}$. The automorphisms of $\mathfrak{g}$ induced by σ, τ are denoted by the same letters σ, τ. There is a decomposition of $\mathfrak{g}$ into direct sums of $+1$, -1-eigenspaces of σ and τ: $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

The subspace $\mathfrak{q}$ can be identified with the tangent space of G/H at the point $x^0 = He$, it is invariant with respect to the adjoint representations $\mathrm{Ad}_\mathfrak{g}$ of H and $\mathrm{ad}_\mathfrak{g}$ of $\mathfrak{h}$.

Now assume, in addition, that G/H is *symplectic*. Then $\mathfrak{h}$ has a non-trivial centre $Z(\mathfrak{h})$. For simplicity we assume that G/H is an orbit $\mathrm{Ad}G \cdot Z_0$ of an element $Z_0 \in \mathfrak{g}$. We can assume that G is simple. Then the statement "G/H is para-Hermitian of the first category" means that the centre $Z(\mathfrak{h})$ of $\mathfrak{h}$ is one-dimensional: $Z(\mathfrak{h}) = \mathbb{R}Z_0$, and Z_0 can be normalized so that the operator $I = (\mathrm{ad}Z_0)_\mathfrak{q}$ is an involution. Therefore, $Z_0 \in \mathfrak{p} \cap \mathfrak{h}$. A symplectic structure on G/H is defined by the bilinear form $\omega(X,Y) = B(X, IY)$ on $\mathfrak{q}$, where $B(X,Y)$ is the Killing form of $\mathfrak{g}$.

The ± 1-eigenspaces $\mathfrak{q}^\pm \subset \mathfrak{q}$ of I are Lagrangian, H-invariant, and irreducible. They are Abelian subalgebras of $\mathfrak{g}$. So $\mathfrak{g}$ becomes a graded Lie algebra:

$$\mathfrak{g} = \mathfrak{q}^- + \mathfrak{h} + \mathfrak{q}^+ (= \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_{+1}).$$

The pair (q^+, q^-) is a Jordan pair [11] with multiplication $\{XYZ\} = \frac{1}{2}[[X,Y],Z]$. Let r and κ be the rank and the genus of this Jordan pair.

Set $Q^\pm = \exp \mathfrak{q}^\pm$. The subgroups $P^\pm = HQ^\pm = Q^\pm H$ are maximal parabolic subgroups of G, with H as a Levi subgroup. One has the following decompositions:

$$G = \overline{Q^+HQ^-} \tag{3.1}$$

$$= \overline{Q^-HQ^+} \tag{3.2}$$

$$= Q^+HK \tag{3.3}$$

$$= Q^-HK, \tag{3.4}$$

where bar means closure and the sets under the bar are open and dense in G. Let us call (3.1) the *Gauss decomposition* and (3.3) the *Iwasawa-type decomposition*. Allowing some slang, let us call (3.2) the *anti-Gauss decomposition* and (3.4) the *anti-Iwasawa-type decomposition*. For an element in G all three factors in (3.1), (3.2) and the first factors in (3.3), (3.4) are defined uniquely, whereas the second and the third factors in (3.3), (3.4) are defined up to an element of $K \cap H$.

For $g \in G$ we define the transformations $\xi \mapsto \tilde{\xi}$ of $\mathfrak{q}^-$ and $\eta \mapsto \hat{\eta}$ of $\mathfrak{q}^+$ taking $\tilde{\xi}$ and $\hat{\eta}$ from the Gauss and the anti-Gauss decompositions:

$$\exp \xi \cdot g = \exp Y \cdot \tilde{h} \cdot \exp \tilde{\xi}, \tag{3.5}$$

$$\exp \eta \cdot g = \exp X \cdot \hat{h} \cdot \exp \hat{\eta}. \tag{3.6}$$

These $\tilde{\xi}$ and $\hat{\eta}$ are defined on open and dense sets in $\mathfrak{q}^-$ and $\mathfrak{q}^+$ respectively, depending on g.

Therefore, G acts on $\mathfrak{q}^- \times \mathfrak{q}^+ : \ (\xi, \eta) \mapsto (\tilde{\xi}, \hat{\eta})$. The stabilizer of $(0,0)$ is $P^+ \cap P^- = H$, so that we obtain the embedding (defined on an open dense set)

$$\mathfrak{q}^- \times \mathfrak{q}^+ \hookrightarrow G/H. \tag{3.7}$$

We may regard (ξ, η) as coordinates in G/H.

The connection between the Gauss and the anti-Gauss decompositions gives us an operator and a function, both very important (see (3.9), (3.10) below). Let $\xi \in \mathfrak{q}^-, \eta \in \mathfrak{q}^+$. Decompose the anti-Gauss product $\exp \xi \cdot \exp(-\eta)$ according to the Gauss decomposition:

$$\exp \xi \cdot \exp(-\eta) = \exp Y \cdot h \cdot \exp X. \tag{3.8}$$

Denote this h by $h(\xi, \eta)$. On $\mathfrak{q}^+$ define the operator

$$K(\xi, \eta) = \mathrm{Ad} h(\xi, \eta)^{-1}|_{\mathfrak{q}^+} \tag{3.9}$$

which is the analogue of the Bergman transform for Hermitian symmetric spaces. In terms of Jordan pairs it becomes:

$$K(\xi, \eta) T = T - 2\{\eta \xi T\} + \{\eta \{\xi T \xi\} \eta\}.$$

Under the action of G the operator $K(\xi, \eta)$ is transformed as follows:

$$K(\tilde{\xi}, \hat{\eta}) = (\mathrm{Ad} \hat{h}^{-1})_{\mathfrak{q}^+} K(\xi, \eta) (\mathrm{Ad} \tilde{h})_{\mathfrak{q}^+},$$

where $\tilde{h}$ and $\hat{h}$ are taken from (3.5) and (3.6).

The function $\det K(\xi, \eta)$ is a polynomial in ξ, η. Moreover, $\det K(\xi, \eta) = N(\xi, \eta)^{\kappa}$, where $N(\xi, \eta)$ is an irreducible polynomial in ξ and η of degree r in ξ and η separately [11]. In view of (3.7) the function

$$b(\xi, \eta) = [\det K(\xi, \eta)]^{-1} \tag{3.10}$$

can be regarded as a function on G/H, becoming an analog of the Bergman kernel. It is invariant with respect to H.

A point x in $G/H \subset \mathfrak{g}$ with coordinates ξ, η is

$$x = Z_0 - X + Z + Y$$

where $X \in \mathfrak{q}^-, Y \in \mathfrak{q}^+$ are given by (3.8), $Z = [\xi, Y] = [\eta, X] \in \mathfrak{h}$, and for X, Y we have equations $X = \xi + AX, Y = -\eta - AY$, where $A = (1/2)\mathrm{ad}[\xi, \eta]$, which allow to find X, Y by means of iterations:

$$X = (E - A)^{-1} \xi, \ Y = -(E + A)^{-1} \eta$$

Let us write a G-invariant metric ds^2, symplectic form ω, measure dx on G/H. Take a basis $E_1, \ldots, E_m$ of $\mathfrak{q}^-$ in such a way that $B(E_i, \tau E_j) = \delta_{ij}$. Then $F_i = \tau E_i$ form a basis of $\mathfrak{q}^+$. Let ξ_i and η_i be the coordinates of $\xi \in \mathfrak{q}^-$ and $\eta \in \mathfrak{q}^+$ in these bases. Then the desired metric, form and measure are:

$$ds^2 = 2 \sum k^{ij}(\xi, \eta) d\xi_i d\eta_j \tag{3.11}$$

$$\omega = 2\sum k^{ij}(\xi,\eta)d\xi_i \wedge d\eta_j \tag{3.12}$$

$$dx = |b(\xi,\eta)|d\xi_1 \dots d\xi_m d\eta_1 \dots d\eta_m \tag{3.13}$$

where k^{ij} are the entries of $K(\xi,\eta)^{-1}$. The function $F(\xi,\eta) = \ln b(\xi,\eta)$ is the potential of the metric:

$$2k^{ij} = \frac{\partial^2 F}{\partial\xi_i \partial\eta_j}.$$

The likeness of (3.11) and (3.12) reflects the fact that G/H has the structure of a manifold over the algebra $D = \{x + jy, x, y \in \mathbb{R}, j^2 = 1\}$.

The coset spaces $S^+ = G/P^-$, $S^- = G/P^+$, $S = K/K \cap H$ are compact manifolds, diffeomorphic to each other by the following correspondence:

$$s^0 k \longleftrightarrow s^{\pm} k,\ k \in K, \tag{3.14}$$

where $s^+ = P^- e, s^- = P^+ e, s^0 = (K \cap H)e$ are the basic points. The natural action of G on $S^{\pm}$ yields two actions of G on S : $s \mapsto \tilde{s}$ and $s \mapsto \hat{s}$, where $s = s^0 k, \tilde{s} = s^0 \tilde{k}, \hat{s} = s^0 \hat{k}$, and $\tilde{k}, \hat{k}$ are obtained from the Iwasawa and the anti-Iwasawa decompositions:

$$kg = \exp Y_1 \cdot \tilde{h}_1 \cdot \tilde{k}, \tag{3.15}$$

$$kg = \exp X_1 \cdot \hat{h}_1 \cdot \hat{k}. \tag{3.16}$$

Set

$$\tilde{s} = s \cdot g; \tag{3.17}$$

then

$$\hat{s} = s \cdot \tau(g). \tag{3.18}$$

The group G acts on $S^- \times S^+$ in a natural way. The stabilizer of the point (s^-, s^+) is H again, so that we obtain the following equivariant embedding $G/H \hookrightarrow S^- \times S^+$. The identification (3.14) gives rise to the equivariant embedding

$$G/H \hookrightarrow S \times S \tag{3.19}$$

where G acts on $S \times S$ by $(s,t) \mapsto (\tilde{s}, \hat{t})$. The image of (3.19) is a single open dense orbit. Denote it by Ω. Thus, $S \times S$ is a compactification of G/H. For the G-orbit structure of $S \times S$, see [9]. Note that G/H can be represented as the tangent bundle of the manifold S.

The spaces $\mathfrak{q}^-$ and $\mathfrak{q}^+$ can be embedded in S:

$$\xi \mapsto s^0 \cdot \exp \xi,\ \eta \mapsto s^0 \cdot \exp \tau(\eta),$$

where $\xi \in \mathfrak{q}^-$, $\eta \in \mathfrak{q}^+$, see (3.17), (3.18), with open dense images; thus either ξ or η can be considered as a coordinate system on S. In these coordinates let us write a K-invariant measure ds on S:

$$ds = \sqrt{b(\xi, \tau\xi)}d\xi \tag{3.20}$$

$$= \sqrt{b(\tau\eta, \eta)}d\eta. \tag{3.21}$$

We now define an important function $\|s,t\|$ on $S \times S$. For $s,t \in S$ take k_s, k_t so that $s = s^0 k_s, t = s^0 k_t$, and apply to $k_s k_t^{-1}$ the Gauss decomposition:

$$k_s k_t^{-1} = \exp Y \cdot h \cdot \exp X. \tag{3.22}$$

It turns out that $\det(\mathrm{Ad}h)_{\mathfrak{q}^+}$ depends only on s,t, but not on the choice of k_s, k_t. We set

$$\|s,t\| = |\det(\mathrm{Ad}h)_{\mathfrak{q}^+}|^{-1/\kappa}, \tag{3.23}$$

where h is taken from (3.22). Formula (3.23) defines $\|s,t\|$ on an open dense subset of $S \times S$. This function is continuous, symmetric and invariant with respect to the diagonal action of K. It can be expanded on the whole $S \times S$, keeping all these properties.

In terms of this function, we can rewrite (3.17) as follows:

$$dx = dx(s,t) = \|s,t\|^{-\kappa} ds\, dt,$$

where $x \mapsto (s,t)$ by (3.19). The orbit Ω is characterized by the condition $\|s,t\| \neq 0$.

The following table contains the list of simple symmetric Lie algebras $\mathfrak{g}/\mathfrak{h}$ that correspond to para-Hermitian symmetric spaces G/H with G simple, see [10]. Here $G_{pq}(\mathbb{F})$ denotes the Grassmann manifold of p-planes in $\mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$; S^{m-1} is the sphere in $\mathbb{R}^m$; $P_2(\mathbb{O})$ denotes the octonion projective plane; $n = p + q$. For aesthetic reasons we denote Lie algebras by capital Latin letters instead of small Gothic ones.

$\mathfrak{g}$	$\mathfrak{h}$	S
$SL(n,\mathbb{R})$	$SL(p,\mathbb{R}) + SL(q,\mathbb{R}) + \mathbb{R}$	$G_{pq}(\mathbb{R})$
$SU^*(2n)$	$SU^*(2p) + SU^*(2q) + \mathbb{R}$	$G_{pq}(\mathbb{H})$
$SU(n,n)$	$SL(n,\mathbb{C}) + \mathbb{R}$	$U(n)$
$SO^*(4n)$	$SU^*(2n) + \mathbb{R}$	$U(2n)/Sp(n)$
$SO(n,n)$	$SL(n,\mathbb{R}) + \mathbb{R}$	$SO(n)$
$SO(p,q)$	$SO(p-1,q-1) + \mathbb{R}$	$(S^{p-1} \times S^{q-1})/\mathbb{Z}_2$
$Sp(n,\mathbb{R})$	$SL(n,\mathbb{R}) + \mathbb{R}$	$U(n)/O(n)$
$Sp(n,n)$	$SU^*(2n) + \mathbb{R}$	$Sp(n)$
$E_{6(6)}$	$SO(5,5) + \mathbb{R}$	$G_{22}(\mathbb{H})/\mathbb{Z}_2$
$E_{6(-26)}$	$SO(1,9) + \mathbb{R}$	$P_2(\mathbb{O})$
$E_{7(7)}$	$E_{6(6)} + \mathbb{R}$	$SU(8)/Sp(4) \cdot \mathbb{Z}_2$
$E_{7(-25)}$	$E_{6(-26)} + \mathbb{R}$	$S^1 \cdot E_6/F_4$

§4. Representations induced from $P^\pm$

For $\mu \in \mathbb{C}$, let ω_μ be the character of H:

$$\omega_\mu(h) = |\det(\mathrm{Ad}h)_{\mathfrak{q}^+}|^{-\mu/\kappa}$$

We restrict ourselves to such characters of H, for simplicity. Extend ω_μ to the character of $P^\pm$, setting it equal to 1 on $Q^\pm$. Consider the representations of G acting on $C^\infty(S)$:

$$T_\mu^\mp = \mathrm{Ind}_{P^\pm}^G \omega_{\pm\mu}$$

In more detail,

$$(T_\mu^-(g)\varphi)(s) = \omega_\mu(\tilde{h}_1)\varphi(\tilde{s}), \quad (T_\mu^+(g)\varphi)(s) = \omega_\mu(\hat{h}_1^{-1})\varphi(\hat{s})$$

we use (3.15), (3.16) and put $s = s^0k, \tilde{s} = s^0\tilde{k}, \hat{s} = s^0\hat{k}$; note that $\omega_\mu(\tilde{h}_1)$ and $\omega_\mu(\hat{h}_1^{-1})$ are well defined because $\omega_\mu(l) = 1$ for $l \in K \cap H$. For the same μ, the representations $T_\mu^\pm$ are connected by τ: $T_\mu^- = T_\mu^+ \circ \tau$, so that if τ is an inner automorphism, then T_μ^+ and T_μ^- are equivalent.

Let (φ, ψ) be the inner product in $L^2(S)$:

$$(\varphi, \psi) = \int_S \varphi(s)\overline{\psi(s)}ds,$$

for ds, see (3.20), (3.21). This Hermitian form is G-invariant for the pairs $(T_\mu^+, T_{-\bar{\mu}-\kappa}^+)$ and $(T_\mu^-, T_{-\bar{\mu}-\kappa}^-)$. Therefore, for $\mathrm{Re}\mu = -\kappa/2$, the representations $T_\mu^\pm$ are unitarizable, and we obtain two *continuous series* of unitary representations.

In a generic case, $T_\mu^\pm$ are irreducible: the reducibility is possible only for real μ satisfying some integrality conditions. Therefore the representations of the continuous series are irreducible for $\mathrm{Im}\mu \neq 0$.

On $C^\infty(S)$ define the operator A_μ:

$$(A_\mu\varphi)(s) = \int_S \|s,t\|^{-\mu-\kappa}\varphi(t)dt;$$

the integral converges absolutely for $\mathrm{Re}\mu < -\kappa + 1$ and is extended on μ-plane as a meromorphic function. This operator intertwines $T_\mu^\pm$ with $T_{-\mu-\kappa}^\mp$:

$$A_\mu T_\mu^\pm = T_{-\mu-\kappa}^\mp A_\mu$$

Moreover,

$$A_{-\mu-\kappa}A_\mu = c(\mu)^{-1}E \tag{4.1}$$

where E is the identity operator and $c(\mu)$ is a meromorphic function.

The representations $T_\mu^\pm$ (degenerate series representations) were studied for separate spaces and with different degree of completeness (for references, see, for example, [16]).

§5. Supercomplete systems and symbols

Let us construct a quantization on G/H (a symbol calculus). The main role belongs to the kernel of the intertwining operator from §4, i.e., to the function

$$\Phi_t(s) = \Phi(s,t) = \|s,t\|^\mu;$$

this function is an analog of Berezin's supercomplete system. The function Φ_t has the reproducing property (which is formula (4.1) written in another form):

$$\varphi(s) = c(\mu)\int_{S\times S} \varphi(\tilde{s})\frac{\Phi(s,\tilde{t})}{\Phi(\tilde{s},\tilde{t})}dx(\tilde{s},\tilde{t})$$

Let $\hat{A}$ be an operator acting on functions on S. Define the *covariant symbol* $A(s,t)$ of $\hat{A}$ as follows:

$$A(s,t) = \frac{(\hat{A}\Phi_t)(s)}{\Phi(s,t)}.$$

We can regard it as a function $A(x)$ on G/H, using (3.19). The operator is recovered by its covariant symbol:

$$(\hat{A}\varphi)(s) = c(\mu)\int_{S\times S} A(s,\tilde{t})\frac{\Phi(s,\tilde{t})}{\Phi(\tilde{s},\tilde{t})}\varphi(\tilde{s})dx(\tilde{s},\tilde{t}). \tag{5.1}$$

The identity operator has 1 as its symbol. The multiplication of operators gives rise to the multiplication of the symbols: $\hat{A}_1\hat{A}_2 = \langle \widehat{A_1 * A_2}\rangle$, where

$$(A_1 * A_2)(s,t) = \int_{S\times S} A_1(s,\tilde{t})A_2(\tilde{s},t)\mathcal{B}(s,t;\tilde{s},\tilde{t})dx(\tilde{s},\tilde{t}), \tag{5.2}$$

$$\mathcal{B}(s,t;\tilde{s},\tilde{t}) = c(\mu)\frac{\Phi(s,\tilde{t})\Phi(\tilde{s},t)}{\Phi(s,t)\Phi(\tilde{s},\tilde{t})}.$$

Let us call this kernel the *Berezin kernel.*

On the other hand, let $\mathring{A}(s,t)$ be a function on $S\times S$. It givs rise to an operator $\hat{A}$ by the formula

$$(\hat{A}\varphi)(s) = c(\mu)\int_{S\times S} \mathring{A}(\tilde{s},\tilde{t})\frac{\Phi(s,\tilde{t})}{\Phi(\tilde{s},\tilde{t})}\varphi(\tilde{s})dx(\tilde{s},\tilde{t})$$

(it differs from (5.1) in the first argument of $\mathring{A}$ only). Let us call the function $\mathring{A}(s,t)$ the *contravariant symbol* of the operator $\hat{A}$. Thus, we have a chain of correspondences: $\mathring{A} \to \hat{A} \to A$. Their composition $\mathcal{B}$, called the *Berezin transform*, links the contra- and covariant symbols by means of the same Berezin kernel:

$$A(s,t) = \int_{S\times S} \mathcal{B}(s,t;\tilde{s},\tilde{t})\mathring{A}(\tilde{s},\tilde{t})dx(\tilde{s},\tilde{t}).$$

Thus, we have a method for constructing a family of algebras $\mathcal{A}_h$: they consist of the covariant symbols $A(s,t) = A(x)$ of operators from some class, the multiplication $*$ in $\mathcal{A}_h$ is given by (5.2), the representations are $A \mapsto \hat{A}$. For the Planck constant we take $h = -d/\mu$ where d depends on normalizations of measures, metrics, etc.

Define the bilinear form $F_\mu(\varphi,\psi)$ on $C^\infty(S)$ by setting

$$F_\mu(\varphi,\psi) = (A_\mu\varphi,\bar{\psi}) = \int_S \|s,t\|^{-\mu-\kappa}\varphi(s)\psi(t)dsdt.$$

Let $\hat{A}'$ be the operator conjugated to an operator $\hat{A}$ with respect to this form: $F_\mu(\hat{A}\varphi,\psi) = F_\mu(\varphi,\hat{A}'\psi)$. Then their symbols are connected by the transposition of the arguments: $A'(s,t) = A(t,s)$. The map $A \mapsto A'$ changes the order of the factors in the product (5.2): $(A_1 * A_2)' = A_2' * A_1'$, so it is an anti-involution of any $\mathcal{A}_h$. In order that CP be in agreement with this anti-involution, we must omit the factor $i = \sqrt{-1}$ in formula (2.1).

By (3.19) the Berezin kernel can be regarded as a function $\mathcal{B}(x,\tilde{x})$ on $G/H \times G/H$. In coordinates ξ,η, it can be written in terms of the function (3.10):

$$\mathcal{B}(x,\tilde{x}) = c(\mu)\left|\frac{b(\xi,\tilde{\eta})b(\tilde{\xi},\eta)}{b(\xi,\eta)b(\tilde{\xi},\tilde{\eta})}\right|^{-\mu/\kappa}$$

where $(\xi,\eta)\mapsto x, (\tilde\xi,\tilde\eta)\mapsto\tilde x$ accordingly to (3.7). In particular (recall that $x^0 = He$ is the basic point of G/H):

$$\mathcal{B}(x,x^0) = c(\mu)|b(\xi,\eta)|^{\mu/\kappa}.$$

The kernel of an intertwining operator depends on a realization of a representation. If we use the coordinates ξ,η, then we must take the function $\Phi(\xi,\eta) = |b(\xi,\eta)|^{-\mu/\kappa}$, in direct analogy with the Hermitian case.

§6. Tensor products

For $\mu\in\mathbb{R}$ the tensor product

$$R_\mu = T^-_{-\mu-\kappa}\otimes T^+_{-\mu-\kappa}$$

acting on $C^\infty(S\times S)$ has the following invariant Hermitian form:

$$E'_\mu(\varphi_1,\varphi_2) = c(\mu)\int \varphi_1(s,t)\overline{\varphi_2(\tilde s,\tilde t)}\left(\|s,\tilde t\|\cdot\|\tilde s,t\|\right)^\mu\, ds\,dt\,d\tilde s\,d\tilde t.$$

The representation R_μ together with E'_μ of G in $C^\infty(S\times S)$ can be considered as an analog of the canonical representation from [18].

Let us restrict this representation to the space $\mathcal{D}(\Omega)$ (the space of C^∞-functions on Ω with compact support). An operator $\varphi\mapsto f$ on $\mathcal{D}(\Omega)$ defined by

$$f(s,t) = \varphi(s,t)\,\|s,t\|^{\mu+\kappa}$$

takes the representation $T^-_{-\mu-\kappa}\otimes T^+_{-\mu-\kappa}$ of G in the representation U of G in $\mathcal{D}(\Omega)$ by translations (see (3.17), (3.18)):

$$U(g)f(s,t) = f\left(s\cdot g, t\cdot\tau(g)\right),$$

and the Hermitian form E'_μ in the Hermitian form E_μ with the Berezin kernel (let us call E_μ the *Berezin form*):

$$E_\mu(f_1,f_2) = \int f_1(s,t)\overline{f_2(\tilde s,\tilde t)}\mathcal{B}(s,t;\tilde s,\tilde t)\,dx(s,t)\,dx(\tilde s,\tilde t), \tag{6.1}$$

or, in terms of G/H:

$$(U(g)f)(x) = f(xg),$$
$$E_\mu(f_1,f_2) = \int f_1(x)\overline{f_2(\tilde x)}\mathcal{B}(x,\tilde x)\,dx\,d\tilde x. \tag{6.2}$$

Thus, we obtain a densely defined G-invariant Hermitian form E_μ on $L^2(G/H)$ (with $\mathcal{D}(G/H)$ as the domain). The integral (6.1), or (6.2), converges absolutely for $\mathrm{Re}\mu > -1$ and is understood as the analytic continuation for other μ's .

We can regard $\mathcal{B}(x,x^0)$ as a H-invariant distribution on G/H. Suppose that we succeed expanding $\mathcal{B}(x,x^0)$ in terms of spherical functions (distributions) on G/H. This is equivalent to writing a Plancherel formula for E_μ. Then we can write expressions of E_μ in terms of Laplace operators $\Delta_1,\ldots,\Delta_r$. This gives us information about the behaviour of E_μ as $\mu\to-\infty$, and we can say whether CP is true.

The representation R_μ on $C^\infty(S \times S)$ is equivalent to the representation U for μ sufficiently near to $\mu = -\kappa/2$. For other μ the decomposition of R_μ contains additional terms so that the space $C^\infty(S \times S)$ needs some "completion" to contain an orthogonal decomposition with respect to E'_μ.

§7. Examples

(*a*) The hyperboloid of one sheet (the imaginary Lobachevsky plane) G/H, where $G = SL(2, \mathbb{R})$, $H = GL(1, \mathbb{R})$, see [13]. The Lie algebra $\mathfrak{g}$ consists of real 2×2 matrices with the zero trace. Let $Z_0 = \mathrm{diag}\{1/2, -1/2\}$. Then H consists of diagonal matrices, $\mathfrak{h} = Z(\mathfrak{h}) = \mathbb{R}Z_0$,

$$\mathfrak{q}^- = \left\{ \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} \right\}, \ \mathfrak{q}^+ = \left\{ \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix} \right\} \tag{7.1}$$

The space G/H consists of matrices

$$x = \frac{1}{2} \begin{pmatrix} x_3 & x_1 - x_2 \\ -x_1 - x_2 & -x_3 \end{pmatrix}$$

satisfying the condition $\det x = -1/4$. In $\mathbb{R}^3$, define a bilinear form $[x, y] = -x_1 y_1 + x_2 y_2 + x_3 y_3$. Then the condition $\det x = -1/4$ is $[x, x] = 1$, i.e., exactly the equation of the hyperboloid of one sheet.

The group G acts on G/H by $x \mapsto g^{-1} x g$ and on $\mathfrak{q}^-$ and $\mathfrak{q}^+$ by fractional linear transformations:

$$\xi \mapsto \frac{\alpha\xi + \gamma}{\beta\xi + \delta}, \ \eta \mapsto \frac{\delta\eta + \beta}{\gamma\eta + \alpha}, \ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G.$$

The embedding (3.7) is

$$x_1 = \frac{\xi + \eta}{1 - \xi\eta}, \ x_2 = \frac{\xi - \eta}{1 - \xi\eta}, \ x_3 = \frac{1 + \xi\eta}{1 - \xi\eta}.$$

The manifold S is the unit circle $|u| = 1$ in $\mathbb{C}$. For this example it is convenient to take the embedding (3.19) as follows: $x \mapsto (u, v), |u| = |v| = 1$, where

$$u = e^{i\alpha} = \frac{x_3 + i x_2}{x_1 + i}, \ v = e^{i\beta} = \frac{x_3 + i x_2}{x_1 - i}$$

(now α, β are not the entries of g). The action of an element of G on u, v is a fractional linear function from $SU(1,1)$, the same for both u, v.

Let us take the measure dx and the Laplace-Beltrami operator Δ on G/H as follows

$$dx = \frac{dx_1 dx_2}{|x_3|} = \frac{2\, d\xi\, d\eta}{(1 - \xi\eta)^2} = \frac{d\alpha\, d\beta}{1 - \cos(\alpha - \beta)},$$

$$\Delta = (1 - \xi\eta)^2 \frac{\partial^2}{\partial\xi\, \partial\eta} = -2\,(1 - \cos(\alpha - \beta))\, \frac{\partial^2}{\partial\alpha\, \partial\beta}.$$

Let U be the unitary representation of G on $L^2(G/H)$ by translations (the quasiregular representation). It decomposes into irreducible unitary representations of three series: the continuous series representations T_σ, $\sigma = -\frac{1}{2} + iu$, $u > 0$, with multiplicity 2, and the discrete series representations $T_n^{\pm}$, $n = 0, 1, 2, \ldots$, with multiplicity 1, see, for instance, [12]. Correspondingly, $L^2(G/H)$ decomposes into the direct sum of four subspaces:

$$L^2(G/H) = L_c^{(0)} + L_c^{(1)} + L_d^+ + L_d^-.$$

Let us write out the expressions for the Berezin transform on these subspaces in terms of Δ:

$$\mathcal{B} = \frac{\Gamma(-\mu+\sigma)\Gamma(-\mu-\sigma-1)}{\Gamma(-\mu)\Gamma(-\mu-1)} \cdot \frac{\sin\mu\pi + (-1)^\varepsilon \sin\sigma\pi}{\sin\mu\pi} \quad \text{on } L_c^{(\varepsilon)},$$

$$\mathcal{B} = \frac{\Gamma(-\mu+\sigma)\Gamma(-\mu-\sigma-1)}{\Gamma(-\mu)\Gamma(-\mu-1)} \qquad \text{on } L_d^{\pm},$$

where the right-hand sides should be considered as functions of $\Delta = \sigma(\sigma+1)$. For L_d^+ and L_d^- we obtain:

$$\mathcal{B} \sim E - \frac{1}{\mu}\Delta \quad (\mu \to -\infty).$$

Thus, CP holds for the discrete spectrum and does not hold for the continuous spectrum. As to algebras with the multiplication (5.2), we can take as such the subspaces of L_d^+ or L_d^- consisting of K-finite vectors. They have no identity element.

(b) The space G/H, where $G = SL(n,\mathbb{R}), H = GL(n-1,\mathbb{R}), n \geq 3$. Here it is more convenient to consider G/H as the orbit of the matrix $x^0 = \mathrm{diag}\{0,\ldots,0,1\}$ under the action $x \mapsto g^{-1}xg$ of G. Then G/H consists of matrices x of rank one and trace one. This space has rank $r = 1$ and genus $\kappa = n$. The spaces of examples (a) and (b) exhaust all para-Hermitian symmetric spaces of rank one up to the covering.

The stabilizer H of x^0 consists of matrices $\mathrm{diag}\{a,b\}$ where $a \in GL(n-1,\mathbb{R})$, $b = (\det a)^{-1}$.

The subalgebras $\mathfrak{q}^-$ and $\mathfrak{q}^+$ consist of matrices of the form (7.1), where ξ is the row $(\xi_1,\ldots,\xi_{n-1})$ and η is the column $(\eta_1,\ldots,\eta_{n-1})$ from $\mathbb{R}^{n-1}$. The embedding (3.7) is

$$x = \frac{1}{1-\xi\eta}\begin{pmatrix} -\eta\xi & -\eta \\ \xi & 1 \end{pmatrix}.$$

In these coordinates on G/H, the Laplace-Beltrami operator is:

$$\Delta = (1-\xi\eta)\sum(\delta_{ij} - \xi_i\eta_j)\frac{\partial^2}{\partial\xi_i\partial\eta_j}.$$

For $x, y \in \mathbb{R}^n$ we write $\langle x,y\rangle = x_1y_1 + \cdots + x_ny_n$ and $|x| = \sqrt{\langle x,x\rangle}$. The manifold S is the unit sphere $S^{n-1} : |s| = 1$ in $\mathbb{R}^n$ with the identification of points s and $-s$, i.e., S is the $(n-1)$-dimensional real projective space. We have $\|s,t\| = |\langle s,t\rangle|$. Any matrix $x \in G/H$ can be written as

$$x = \frac{t's}{\langle t,s\rangle},$$

where $t,s \in S^{n-1}, \langle t,s\rangle \neq 0$, the prime denotes matrix transposition. Let ds be the Euclidean measure on S^{n-1}. The following measure dx on G/H is G-invariant:

$$dx = |\langle t,s\rangle|^{-n}dt\,ds.$$

The supercomplete system is $\Phi(s,t) = |\langle s,t\rangle|^\mu$. In terms of G/H the Berezin kernel is:

$$\mathcal{B}(x,\tilde{x}) = c(\mu)|\mathrm{tr}(x\tilde{x})|^\mu,$$

where

$$c(\mu) = \left\{2^{n+1}\pi^{n-2}\Gamma(-\mu-n+1)\Gamma(\mu+1)[\cos(\mu+\frac{n}{2})\pi - \cos\frac{n\pi}{2}]\right\}^{-1}.$$

The quasiregular representation U of G on G/H decomposes into irreducible unitary representations of two series: the continuous series representations $T_{\sigma,\varepsilon}, \sigma = (1/2)(1-n)+iu, u > 0, \varepsilon = 0,1$, and the discrete series representations $T_{\sigma(m)}, \sigma(m) = (1/2)(2-n)+m, m = 0,1,2,\ldots$; all with multiplicity 1, see [14], [15], [8]. Let us write the expressions of the Berezin form ($\mu < (1-n)/2$) in terms of Δ:

$$\mathcal{B} = \frac{\Gamma(-\mu+\sigma)\Gamma(-\mu-\sigma-n+1)}{\Gamma(-\mu)\Gamma(-\mu-n+1)} \cdot \frac{\cos\mu\pi + (-1)^\varepsilon\cos\sigma\pi}{\cos\mu\pi + 1} \quad (n \text{ odd}),$$

$$\mathcal{B} = \frac{\Gamma(-\mu+\sigma)\Gamma(-\mu-\sigma-n+1)}{\Gamma(-\mu)\Gamma(-\mu-n+1)} \cdot \frac{\sin\mu\pi + (-1)^\varepsilon\sin\sigma\pi}{\sin\mu\pi} \quad (n \text{ even}).$$

The right-hand sides should be regarded as functions of $\Delta = \sigma(\sigma+n-1)$. In both formulae the first fraction behaves as $1-\mu^{-1}\Delta$ when $\mu \to -\infty$. It is just what we need for CP. In the second fractions, the term with $(-1)^\varepsilon$ disappears on the discrete spectrum for n even. So we have CP on the discrete spectrum for n even.

We can unite both formulae above

$$\mathcal{B} = (-1)^\varepsilon \frac{\Gamma(\frac{-\mu+\sigma+\varepsilon}{2})\Gamma(\frac{-\mu-\sigma-n+1+\varepsilon}{2})}{\Gamma(\frac{\mu-\sigma+1+\varepsilon}{2})\Gamma(\frac{\mu+\sigma+n+\varepsilon}{2})} \cdot \frac{\Gamma(\frac{\mu+1}{2})\Gamma(\frac{\mu+n}{2})}{\Gamma(-\frac{\mu}{2})\Gamma(\frac{-\mu-n+1}{2})}$$

For the decomposition of tensor product R_μ acting on $C^\infty(S\times S)$, see [6,7]. For $\mu > (-n+1)/2$ additional representations act on distributions on $S\times S$ concentrated on the boundary Γ of Ω. This action is diagonalizable (the corresponding representation decomposes into the direct sum of irreducible representations). In general, the appearance of representations acting on distributions concentrated on manifolds of lower dimension is one of the intriguing phenomena in harmonic analysis.

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